Stokes Structure and Direct Image of Irregular Singular $\mathcal{D}$-Modules

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Abstract: In this paper we will present a way of examining the Stokes structure of certain irregular singular $\mathcal{D}$-modules, namely the direct image of exponentially twisted regular singular meromorphic connections, in a topological point of view. This topological description enables us to compute Stokes data for an explicit example concretely.

1 Introduction – Preliminaries

Recently there was a lot of progress in proving an irregular Riemann-Hilbert correspondence in higher dimensions. Sabbah introduced the notion of *good* meromorphic connections and proved an equivalence of categories in this case [Sab13], which was extended to the general case by Mochizuki [Moc09] and Kedlaya [Ked10]. Furthermore d'Agnolo/Kashiwara proved an irregular Riemann-Hilbert correspondence for all dimensions using subanalytic sheaves [DK].
Nevertheless it is still difficult to describe the Stokes phenomenon for explicit situations and to calculate Stokes data concretely. In their recent article, Hien/Sabbah developed a topological way to determine Stokes data of the Laplace transform of an elementary meromorphic connection [HS]. The techniques used by Hien/Sabbah can be adapted to other situations. Hence in this article we will present a topological view of the Stokes phenomenon for the direct image of an exponentially twisted meromorphic connection $\mathcal{M}$ in a 2-dimensional complex manifold. Namely we will consider the following situation:
Let $X = \Delta \times \mathbb{P}^1$ be a complex manifold, where Δ denotes an open disc in $0 \in \mathbb{C}$ with coordinate t. We denote the coordinate of $\mathbb{P}^1$ in 0 by x and the coordinate in ∞ by $y = \frac{1}{x}$. Let $\mathcal{M}$ be a regular singular holonomic $\mathcal{D}_X$-module. We have the following projections:

$$\begin{tikzcd} & \Delta \times \mathbb{P}^1 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \Delta & & \mathbb{P}^1 \end{tikzcd}$$

Let $\overline{D}$ denote the singular locus of $\mathcal{M}$, which consists of $\{0\} \times \mathbb{P}^1 =: D$, $\Delta \times \{\infty\}$ and some additional components. We will distinguish between the components

- $S_{i\in I}$ $(I = \{1, \dots, n\})$, which meet D in the point $(0, \infty)$ and
- $\widetilde{S}_{j\in J}$ $(J = \{1, \dots, m\})$, which meet $\{0\} \times \mathbb{P}^1$ in some other point.

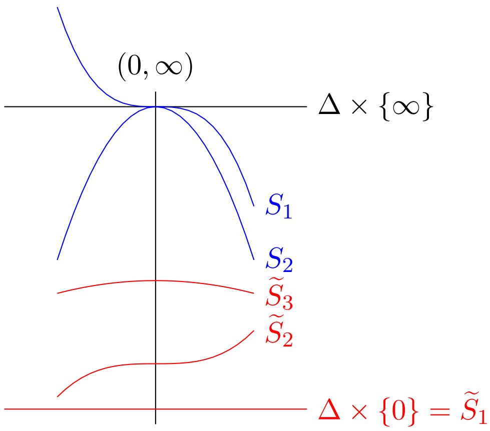
Furthermore we will require the following conditions on $\overline{D}$:

Assumption 1.1: *Locally in $(0, \infty)$ the irreducible components S_i of the divisor $\overline{D}$ achieve the following conditions:*

- *$S_i : \mu_i(t)\, y = t^{q_i}$, where μ_i is holomorphic and $\mu_i(0) \neq 0$.*
- *For $i \neq j$ either $q_i \neq q_j$ or $\mu_i(0) \neq \mu_j(0)$ holds.*

Assumption 1.2: *The irreducible components $\widetilde{S}_j$ intersect D in pairwise distinct points. Moreover we assume $\widetilde{S}_j$ to be smooth, i. e. locally around the intersection point they can be described as*

$$\widetilde{S}_j : \mu_j(t)\, x = t^{q_j}.$$



We want to examine $p_+(\mathcal{M} \otimes \mathcal{E}^q)$. This is a complex with $\mathcal{H}^k p_+(\mathcal{M} \otimes \mathcal{E}^q) = 0$ for $k \neq -1, 0$. Furthermore one can show that even $\mathcal{H}^{-1} p_+(\mathcal{M} \otimes \mathcal{E}^q) = 0$ on Δ^* (cf. [Sab08], p. 161), i. e. it is only supported in 0. Therefore, we will consider $\mathcal{H}^0 p_+(\mathcal{M} \otimes \mathcal{E}^q)$. We will assume Δ small enough such that 0 is the only singularity of the $\mathcal{D}_\Delta$-module $\mathcal{H}^0 p_+(\mathcal{M} \otimes \mathcal{E}^q)$ and we denote its germ at 0 by

$$\mathcal{N} := \left(\mathcal{H}^0 p_+(\mathcal{M} \otimes \mathcal{E}^q)\right)_0.$$

In the following we will take a closer look at the Stokes-filtered local system $(\mathcal{L}, \mathcal{L}_{\leq \psi})$ (Chapter 2), which is associated to the $\mathcal{D}_\Delta$-module $\mathcal{N}$ by the irregular Riemann-Hilbert correspondence

as mentioned above. We will use an isomorphism

$$\Omega : \mathcal{L}_{\leq\psi} \xrightarrow{\cong} \mathcal{H}^1 R\widetilde{p}_* \,\mathrm{DR}^{mod\, D}\left(\mathcal{M} \otimes \mathcal{E}^{\frac{1}{y}} \otimes \mathcal{E}^{-\psi}\right)$$

(proved by Mochizuki) to develop a topological description for $\mathcal{L}_{\leq\psi}$.

In Chapter 3 we will use this topological perspective to present a way of determining Stokes matrices for an explicit example, where the singular locus of a meromorphic connection $\mathcal{M}$ of rank r only consists of two additional irreducible components, namely $(S_1 : y = t)$ and $(\widetilde{S}_1 : x = 0)$. We will describe the Stokes-filtered local system $\mathcal{L}$ (which in this case will be of exponential type) in terms of linear data, namely a set of linear Stokes data of exponential type, defined as follows:

Let $\Phi = \{\phi_i \mid i \in I\}$ denote a finite set of exponents ϕ_i of pole order ≤ 1 and let $\theta_0 \in \mathbb{S}^1$ be a generic angle, i. e. it is no Stokes direction with respect to the ϕ_is. We get a unique ordering of the exponents $\phi_0 <_{\theta_0} \phi_1 <_{\theta_0} \ldots <_{\theta_0} \phi_n$ and the reversed ordering for $\theta_1 := \theta_0 + \pi$.

Definition 1.3 ([HS11], Def 2.6)**:** The *category of Stokes data of exponential type* (for a set of exponents Φ ordered by θ_0) has objects consisting of two families of $\mathbb{C}$-vector spaces (G_{ϕ_i}, H_{ϕ_i}) and two morphisms

$$\bigoplus_{i=0}^{n} G_{\phi_i} \xrightarrow{S} \bigoplus_{i=0}^{n} H_{\phi_i} \qquad \bigoplus_{i=0}^{n} H_{\phi_{n-i}} \xrightarrow{S'} \bigoplus_{i=0}^{n} G_{\phi_{n-i}}$$

such that

1. S is a block upper triangular matrix, i. e. $S_{ij} : G_{\phi_i} \to H_{\phi_j}$ is zero for $i > j$ and S_{ii} is invertible (thus S is invertible and dim $G_{\phi_i} =$ dim H_{ϕ_i})

2. S' is a block lower triangular matrix, i. e. $S'_{ij} : H_{\phi_{n-i}} \to G_{\phi_{n-j}}$ is zero for $i < j$ and S'_{ii} is invertible (thus S' is invertible)

A morphism consists of morphisms of $\mathbb{C}$-vector spaces $\lambda_i^G : G_{\phi_i} \to G'_{\phi_i}$ and $\lambda_i^H : H_{\phi_i} \to H'_{\phi_i}$, which are compatible with the corresponding diagrams.

The correspondence between Stokes-filtered local systems of exponential type and linear Stokes data is stated in the following theorem. For a proof we refer to [HS11], p. 12/13.

Theorem 1.4: *There is an equivalence of categories between the Stokes-filtered local systems of exponential type and Stokes data of exponential type.*

By associating the Stokes-filtered local system to a set of Stokes data via this equivalence of categories and using the isomorphism Ω we will finally get an explicit description of the Stokes

data in our concrete example. It is stated in the following

Theorem 1.5: *Fix the following data:*

- $L_0 = \mathbb{V} \oplus \mathbb{V}$, $L_1 = \mathbb{V} \oplus \mathbb{V}$
- $S_0^1 = N_\pi = \begin{pmatrix} -1 & 1 - ST^{-1} \\ 0 & -ST^{-1} \end{pmatrix}$, $S_1^0 = (\mu_0^\pi \circ \mu_\pi^0) \cdot N_0 = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \cdot \begin{pmatrix} -TS^{-1} & 0 \\ 1 - TS^{-1} & -1 \end{pmatrix}$

where $\mathbb{V}$ is the generic stalk of the local system attached to $\mathcal{M}$ and S, T denote the monodromies around the strict transforms of the irreducible components $\widetilde{S}_1$, S_1 in the singular locus of $\mathcal{M}$ and U denotes the mondromy around the component $\{0\} \times \mathbb{P}^1$. Then

$$\left(L_0, L_1, S_0^1, S_1^0\right)$$

defines a set of Stokes data for $\mathcal{H}^0 p_+ \left(\mathcal{M} \otimes \mathcal{E}^{\frac{1}{y}}\right)$.

2 Stokes-filtered local system $(\mathcal{L}, \mathcal{L}_{\leq \psi})$

In [Rou07] Roucairol examined the formal decomposition of direct images of $\mathcal{D}$-modules in the situation presented above. She determined the exponential factors as well as the rank of the corresponding regular parts appearing in the formal decomposition of $\mathcal{N}$. In our case this leads to the following result:

Theorem 2.1 ([Rou07], Thm 1)**:** *Let $\mathcal{M}$ be a regular singular $\mathcal{D}_X$-module with singular locus $\overline{D}$ which achieves the previous assumptions 1.1 and 1.2. Then $\hat{\mathcal{N}} := \left(\mathcal{H}^0 p_+ \left(\mathcal{M} \otimes \mathcal{E}^q\right)\right)_0^\wedge$ decomposes as*

$$\hat{\mathcal{N}} = R_0 \oplus \bigoplus_{i \in I} \left(R_i \otimes \mathcal{E}^{\psi_i(t)}\right),$$

where R_0, R_i $(i \in I)$ are regular singular $\mathcal{D}_\Delta$- modules and $\psi_i(t) = \mu_i(t)\, t^{-q_i}$. Moreover we have:

- $rk\,(R_i) = \dim \Phi_{P_i}$ *where Φ_{P_i} denotes the vanishing cycles of* $\mathrm{DR}(e^+\mathcal{M})$ *at the intersection point P_i of (a strict transform of) S_i with the exceptional divisor after a suitable blow up e.*
- $rk\,(R_0) = \sum_{j \in J} \dim\, \Phi_{\widetilde{P}_j}$, *where $\Phi_{\widetilde{P}_j}$ denotes the vanishing cycles of* $\mathrm{DR}(\mathcal{M})$ *at the intersection point $\widetilde{P}_j$ of $\widetilde{S}_j$ with D.*

Proof: The formal decomposition and the statement about the rank of the R_i is exactly Roucairol's theorem applied to our given situation. With the same arguments as in Roucairol's proof one can also show that $rk\,(R_0) = \sum_{j \in J} \dim\, \Phi_{\widetilde{P}_j}$. □

Identify $\mathbb{S}^1 = \{\vartheta \mid \vartheta \in [0, 2\pi)\}$ and denote $\mathcal{P} := x^{-1}\mathbb{C}[x^{-1}]$. Let us recall the following definition of a Stokes-filtered local system.

Definition 2.2 ([Sab13], Lemma 2.7)**:** Let $\mathcal{L}$ be a local system of $\mathbb{C}$-vector spaces on $\mathbb{S}^1$. We will call $(\mathcal{L}, \mathcal{L}_{\leq})$ a *Stokes-filtered local system*, if it is equipped with a family of subsheaves $\mathcal{L}_{\leq\phi}$ (indexed by $\phi \in \mathcal{P} := x^{-1}\mathbb{C}[x^{-1}]$) satisfying the following conditions:

1. For all $\vartheta \in \mathbb{S}^1$, the germs $\mathcal{L}_{\leq\phi,\vartheta}$ form an exhaustive increasing filtration of $\mathcal{L}_\vartheta$
2. $gr_\phi\mathcal{L} := \mathcal{L}_{\leq\phi}/\mathcal{L}_{<\phi}$ is a local system on $\mathbb{S}^1$ (where $\mathcal{L}_{<\phi,\vartheta} := \sum_{\psi<_\vartheta\phi} \mathcal{L}_{\leq\psi,\vartheta}$)
3. $\dim\, \mathcal{L}_{\leq\phi,\vartheta} = \sum_{\psi\leq_\vartheta\phi} \dim\, gr_\psi\mathcal{L}_\vartheta$

Assume that $\mathcal{M}$ (and therefore the direct image $\mathcal{N}$ defined above) is a meromorphic connection. Let $\mathcal{L}'$ denote the local system on Δ^* corresponding to the meromorphic connection $\mathcal{N}$. Moreover let $\pi : \widetilde{\Delta} \to \Delta, (r, e^{i\vartheta}) \mapsto r \cdot e^{i\vartheta}$ be the real oriented blow up of Δ in the singularity 0 and $j : \Delta^* \hookrightarrow \widetilde{\Delta}$. Consider $j_*\mathcal{L}'$ and restrict it to the boundary $\partial\Delta$. We get a local system on $\mathbb{S}^1 \cong \partial\widetilde{\Delta}$ and define $\mathcal{L} := j_*\mathcal{L}'_{|\mathbb{S}^1}$. For $\phi \in \mathcal{P}$ define

$$\mathcal{L}_{\leq\phi} := \mathcal{H}^0\ \mathrm{DR}^{mod\,0}_{\partial\widetilde{\Delta}}(\mathcal{N} \otimes \mathcal{E}^{-\phi}) \text{ and } \mathcal{L}_{<\phi} := \mathcal{H}^0\ \mathrm{DR}^{<0}_{\partial\widetilde{\Delta}}(\mathcal{N} \otimes \mathcal{E}^{-\phi}),$$

where $\mathrm{DR}^{mod\,0}$ resp. $\mathrm{DR}^{<0}$ denotes the moderate (resp. rapid decay) de Rham complex. According to the equivalence of categories between germs of a $\mathcal{O}_\Delta(*0)$-connection and Stokes-filtered local systems stated by Deligne/Malgrange (cf. [Mal91]), $(\mathcal{L}, \mathcal{L}_{\leq})$ forms the Stokes-filtered local system associated to $\mathcal{N}$.
Moreover by the Hukuhara-Turrittin-Theorem (cf. [Sab07], p. 109) the formal decomposition of $\mathcal{N}$ can be lifted locally on sectors to $\partial\widetilde{\Delta} = \mathbb{S}^1$. Thus to determine the filtration $\mathcal{L}_{\leq}$, it is enough to consider the set of exponential factors (respectively their polar part) appearing in the formal decomposition, since the moderate growth property of the solutions of an elementary connection $\mathcal{R} \otimes \mathcal{E}^\phi$ ($\phi \in \mathcal{P}$) only depends on the asymptotical behavior of e^ϕ. We denote the set of exponential factors of the formal decomposition of $\mathcal{N}$ by $\mathcal{P}_\mathcal{N} := \{\psi_i \mid i \in I_0\}$ whereby $I_0 := I \cup \{0\}$ and $\psi_0 := 0$.

Definition 2.3: For $\vartheta \in \mathbb{S}^1$ we define the following ordering on $\mathcal{P}$: $\phi \leq_\vartheta \psi :\Leftrightarrow e^{\phi-\psi} \in \mathcal{A}^{mod\,0}$.

Remark 2.4: 1. For $\psi_i, \psi_j \in \mathcal{P}_\mathcal{N}$ appearing in the formal decomposition of $\mathcal{N}$ we can determine $\vartheta \in \mathbb{S}^1$, such that $\psi_i \leq_\vartheta \psi_j$ (cf. [Sab13], ex. 1.6):

$$\psi_i \leq_\vartheta \psi_j \Leftrightarrow \psi_i - \psi_j \leq_\vartheta 0 \Leftrightarrow \mu_i\,(t)\,t^{-q_i} - \mu_j\,(t)\,t^{-q_j} \leq_\vartheta 0.$$

There exists a finite set of $\vartheta \in \mathbb{S}^1$, where ψ_i and ψ_j are not comparable, i. e. neither

$\psi_i \leq_\vartheta \psi_j$ nor $\psi_j \leq_\vartheta \psi_i$ holds. We call these angles the *Stokes directions* of (ψ_i, ψ_j).

2. For $\vartheta_0 \in \mathbb{S}^1$ not being a Stokes direction of any pair (ψ_i, ψ_j) in $\mathcal{P}_\mathcal{N}$ we get a total ordering of the ψ_is with respect to ϑ_0: $\psi_{\tilde{0}} <_{\vartheta_0} \ldots <_{\vartheta_0} \psi_{\tilde{n}}$.

Corollary 2.5: *For $\vartheta \in \mathbb{S}^1$ we get the following statement:*

1. $\psi = 0$*:*

$$\dim\left(\mathcal{L}_{\leq 0}\right)_\vartheta = \sum_{j \in J} \Phi_{\widetilde{P}_j} + \sum_{\{i | \psi_i \leq_\vartheta 0\}} \Phi_{P_i}$$

2. $\psi \neq 0$*,* $\vartheta \in \left(\frac{\frac{\pi}{2} + arg(-\mu(0))}{q}, \frac{\frac{3\pi}{2} + arg(-\mu(0))}{q}\right)$ *mod* $\frac{2\pi}{q}$*:*

$$\dim\left(\mathcal{L}_{\leq \psi}\right)_\vartheta = \sum_{j \in J} \Phi_{\widetilde{P}_j} + \sum_{\{i | \psi_i \leq_\vartheta \psi\}} \Phi_{P_i}$$

3. $\psi \neq 0$*,* $\vartheta \in \left(\frac{-\frac{\pi}{2} + arg(-\mu(0))}{q}, \frac{\frac{\pi}{2} + arg(-\mu(0))}{q}\right)$ *mod* $\frac{2\pi}{q}$*:*

$$\dim\left(\mathcal{L}_{\leq \psi}\right)_\vartheta = \sum_{\{i | \psi_i \leq_\vartheta \psi\}} \Phi_{P_i}$$

Proof: This is a direct consequence of the formal decomposition and the fact that for $\psi \neq 0$ we have

$$\begin{aligned} 0 \leq_\vartheta \psi \Leftrightarrow -\mu(t)\, t^{-q} \leq_\vartheta 0 &\Leftrightarrow arg\left(-\mu(0)\right) - q\vartheta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ &\Leftrightarrow \vartheta \in \left(\frac{\frac{\pi}{2} + arg\left(-\mu\left(0\right)\right)}{q}, \frac{\frac{3\pi}{2} + arg\left(-\mu\left(0\right)\right)}{q}\right) mod\ \frac{2\pi}{q} \end{aligned}$$

□

2.1 Topological description of the stalks

As before we assume that $\mathcal{M}$ is a meromorphic connection with regular singularities along its divisor. Our aim is to determine the Stokes structure of $\mathcal{N}$ by using a topological point of view, i. e. we will develop a topological description of the Stokes-filtered local system $(\mathcal{L}, \mathcal{L}_\leq)$. Therefore we will use the following theorem:

Theorem 2.6 ([Moc], Cor. 4.7.5)**:** *There is an isomorphism*

$$\Omega : \mathcal{L}_{\leq\psi} := \mathcal{H}^0\, \mathrm{DR}^{mod\, 0}_{\widetilde{\Delta}} \left(\mathcal{N} \otimes \mathcal{E}^{-\psi}\right) \to \widetilde{\mathcal{L}}_{\leq\psi} := \mathcal{H}^1 R\widetilde{p}_*\, \mathrm{DR}^{mod\, D}_{\widetilde{X}(D)} \left(\mathcal{M} \otimes \mathcal{E}^{\frac{1}{y}} \otimes \mathcal{E}^{-\psi}\right)$$

Here $\widetilde{X}(D)$ denotes the real-oriented blow up of X along the divisor component D, $\mathrm{DR}^{mod\, D}(\mathcal{M})$ denotes the moderate de Rham complex of a meromorphic connection $\mathcal{M}$ on X and $\widetilde{p} : \widetilde{X}(D) \to \widetilde{\Delta}$ corresponds to the projection p in the real- oriented blow up space $\widetilde{X}(D)$ along D.

Theorem 2.6 is a special case of [Moc], Cor. 4.7.5. In the following we will develop a topological view of the right hand side of the above isomorphism, which enables us to describe the Stokes-filtered local system $(\mathcal{L}, \mathcal{L}_{\leq})$ more explicitly.
First let us examine the behavior of $\widetilde{\mathcal{L}}_{\leq\psi}$ with respect to birational maps.

Proposition 2.7: *Let $e : Z \to \Delta \times \mathbb{P}^1$ a birational map (i.e. a sequence of point blow-ups), $g(t,y) := \frac{1}{y} - \psi(t)$, $D_Z = e^{-1}(D)$. Then:*

$$\mathrm{DR}^{mod\, D}_{\widetilde{X}(D)}\left(\mathcal{M} \otimes \mathcal{E}^{g(t,y)}\right) \cong R\tilde{e}_* \,\mathrm{DR}^{mod\, D_Z}_{\widetilde{Z}(D_Z)}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right)$$

Proof: We know that e is proper. Furthermore we assumed that $\mathcal{M}$ is a meromorphic connection with regular singularities along D, i.e. particularly that $\mathcal{M} \otimes \mathcal{E}^{g(t,y)}$ is a holonomic $\mathcal{D}_X$-module and localized along D. Thus, using the fact that $e_+\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right) \cong \mathcal{M} \otimes \mathcal{E}^{g(t,y)}$, we can apply [Sab13], Prop. 8.9:

$$\begin{aligned}\mathrm{DR}^{mod\, D}\left(\mathcal{M} \otimes \mathcal{E}^{g(t,y)}\right) &\cong \mathrm{DR}^{mod\, D}\left(e_+\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right)\right)\\ &\cong R\tilde{e}_* \,\mathrm{DR}^{mod\, D_Z}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right)\end{aligned}$$

□

Proposition 2.8: *Denote $D' := D \cup (\Delta \times \{\infty\})$. Let $D'_Z := e^{-1}(D')$ and $\overline{D}_Z := e^{-1}(\overline{D})$. If $\overline{D}_Z$ is a normal crossing divisor, we have isomorphisms*

$$R\tilde{e}_* \,\mathrm{DR}^{mod\, D_Z}_{\widetilde{Z}(D_Z)}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right) \cong R\tilde{e}_* \,\mathrm{DR}^{mod\, D'_Z}_{\widetilde{Z}(D'_Z)}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right)$$

and

$$R\tilde{e}_* \,\mathrm{DR}^{mod\, D_Z}_{\widetilde{Z}(D_Z)}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right) \cong R\tilde{e}_* \,\mathrm{DR}^{mod\, \overline{D}_Z}_{\widetilde{Z}(\overline{D}_Z)}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right)$$

where $\tilde{e}$ denotes the induced map in the particular blow up spaces.

Proof: Assume $\overline{D}_Z$ a normal crossing divisor and consider the identity map $Z \xrightarrow{Id} Z$, which obviously induces an isomorphism on $Z \setminus \overline{D}_Z \to Z \setminus \overline{D}_Z$. We obtain 'partial' blow up maps

$$\widetilde{Id}_1 : \widetilde{Z}\left(D'_Z\right) \to \widetilde{Z}\left(D_Z\right) \text{ and } \widetilde{Id}_2 : \widetilde{Z}\left(\overline{D}_Z\right) \to \widetilde{Z}\left(D_Z\right)$$

which induce the requested isomorphisms. These are variants of Prop. 8.9 in [Sab13] (see also [Sab13], Prop. 8.7 and Rem. 8.8). □

Corollary 2.9:

$$\mathcal{H}^1\left(R\widetilde{p}_* \mathrm{DR}^{mod\,D}\left(\mathcal{M}\otimes\mathcal{E}^{g(t,y)}\right)\right)\cong\mathcal{H}^1\left(R\left(\widetilde{p\circ e}\right)_* \mathrm{DR}^{mod(\star)}\left(e^+\mathcal{M}\otimes\mathcal{E}^{g(t,y)\circ e}\right)\right)$$

where $\star = D_Z$ *(resp.* D'_Z*, resp.* $\overline{D}_Z$*).*

As $(\mathcal{L},\mathcal{L}_{\leq})$ was defined on $\partial\widetilde{\Delta}$ we will restrict our investigation to the boundaries of the relevant blow up spaces, i. e. we consider $\partial\widetilde{\Delta}$, $\partial\widetilde{X}\left(D\right)$, $\partial\widetilde{Z}\left(D_Z\right)$, $\partial\widetilde{Z}\left(D'_Z\right)$ and $\partial\widetilde{Z}(\overline{D}_Z)$. We will take a closer look at the stalks:

Lemma 2.10: *Let* $\vartheta\in\mathbb{S}^1$*. There is an isomorphism*

$$\left(\mathcal{H}^1\left(R\left(\widetilde{p\circ e}\right)_* \mathrm{DR}^{mod(D_Z)}\left(e^+\mathcal{M}\otimes\mathcal{E}^{g(t,y)\circ e}\right)\right)\right)_\vartheta\cong \mathbb{H}^1\left(\left(\widetilde{p\circ e}\right)^{-1}(\vartheta),\mathrm{DR}^{mod(D_Z)}\left(e^+\mathcal{M}\otimes\mathcal{E}^{g(t,y)\circ e}\right)\right)$$

The same holds for D'_Z *and* $\overline{D}_Z$ *instead of* D_Z*.*

Proof: $\widetilde{p\circ e}$ is proper, thus the claim follows by applying the proper base change theorem (cf. [Dim04], Th. 2.3.26, p. 41). □

In the following we will consider

$$\mathcal{F}_\psi := \mathrm{DR}^{mod\left(D'_Z\right)}\left(e^+\mathcal{M}\otimes\mathcal{E}^{g(t,y)\circ e}\right)$$

on the fiber $\left(\widetilde{p\circ e}\right)^{-1}(\vartheta)$ in $\partial\widetilde{Z}\left(D'_Z\right)$, which in our case will be a 1-dimensional complex analytic space. Therefore we can assume $\mathcal{F}_\psi$ to be a perverse sheaf, i.e. a 2-term complex $\mathcal{F}_\psi : 0\to\mathcal{F}^0\to\mathcal{F}^1\to 0$ with additional conditions on the cohomology sheaves $\mathcal{H}^0\left(\mathcal{F}_\psi\right)$ and $\mathcal{H}^1\left(\mathcal{F}_\psi\right)$. Obviously $\mathcal{H}^i\left(\mathcal{F}_\psi\right)=0$ for $i\neq 0,1$ (cf. [Dim04], Ex. 5.2.23, p. 139).
Furthermore we can restrict $\mathcal{F}_\psi$ to the set $B^\vartheta_\psi := \left\{\zeta\in\left(\widetilde{p\circ e}\right)^{-1}(\vartheta) \mid \left(\mathcal{H}^\bullet(\mathcal{F}_\psi)\right)_\zeta\neq 0\right\}$, which is an open subset of $\left(\widetilde{p\circ e}\right)^{-1}(\vartheta)$. We denote the open embedding by $\beta^\vartheta_\psi : B^\vartheta_\psi\hookrightarrow\left(\widetilde{p\circ e}\right)^{-1}(\vartheta)$. Thus by interpreting $\mathcal{F}_\psi$ as a complex of sheaves on B^ϑ_ψ, we have to compute

$$\mathbb{H}^1\left(\left(\widetilde{p\circ e}\right)^{-1}(\vartheta),\beta^\vartheta_{\psi,!}\mathcal{F}_\psi\right)\cong\mathbb{H}^1_c\left(B^\vartheta_\psi,\mathcal{F}_\psi\right).$$

2.1.1 Construction of a Resolution of Singularities

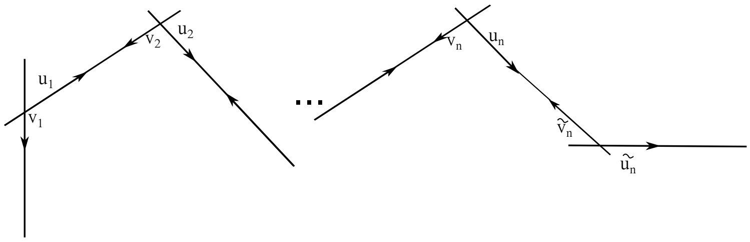
In the following sections we will construct a suitable blow up map e, such that we can describe $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ and B_{ψ}^{ϑ} more concretely.

Lemma 2.11: *Let $g(t,y) = \frac{1}{y} - \psi(t)$. There exists a sequence of blow up maps e such that*

1. *$g \circ e$ holomorphic or good, i. e. $(g \circ e)(u,v) = \frac{1}{u^m v^n}\beta(u,v)$, whereby β holomorphic and $\beta(0,v) \neq 0$.*
2. *For all i the strict transform of S_i intersects D_Z in a unique point P_i.*

Proof: The divisor components S_i are given by $S_i : \mu_i(t)\, y = t^{q_i}$. Let $n := max\{q_i\}$. We distinguish two cases:

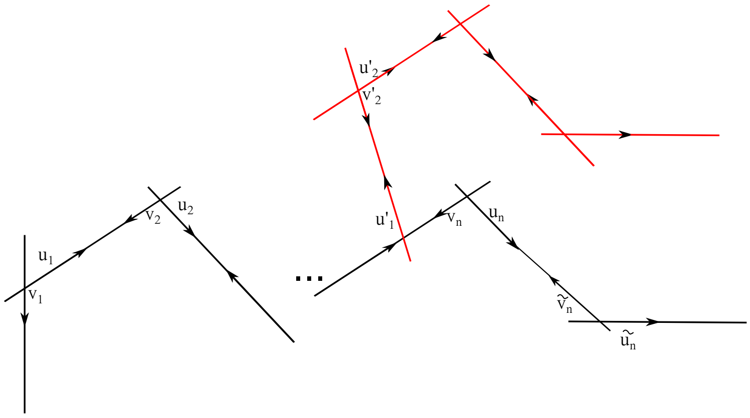
1. $\psi = 0$, i. e. $g(t,y) = \frac{1}{y}$. After n blow ups in $(0,0)$ we get an exceptional divisor D_Z with local coordinates
$$t = u_k v_k, y = u_k^{k-1} v_k^k \text{ and } t = u_{\tilde{n}}, y = \tilde{u}_n^n \tilde{v}_n$$
and $g \circ e$ is good or holomorphic in every point.



The intersection points P_i of the strict transform of $S_i : \mu_i(t)\, y = t^{q_i}$ is given by
$$P_i = \left(0, \frac{1}{\mu_i(0)}\right)$$
in the proper coordinates (i. e. (u_{q_i+1}, v_{q_i+1}) for $q_i < n$ and $(\tilde{u}_n, \tilde{v}_n)$ for $q_i = n$)

2. $\psi \neq 0$: ψ is given by $\psi(t) = \mu(t)\, t^{-q}$, so $g(t,y) = \frac{t^q - \mu(t) y}{y t^q}$.$(g \circ e)$ is good in every point except:
   - $q < n, k = q+1 : P = \left(0, \frac{1}{\mu(0)}\right)$ with local coordinates (u_{q+1}, v_{q+1})
   - $q = n, k = n : P = \left(0, \frac{1}{\mu(0)}\right)$ with local coordinates $(\tilde{u}_n, \tilde{v}_n)$

   Let $q < n$. After changing coordinates $u' = u_k, v' = v_k - \frac{1}{\mu(u_k v_k)}$ and after q blow-ups in $(0,0)$, $(g \circ e)$ is good for every point of D_Z in local coordinates (u_s', v_s') and holomorphic for every point of D_Z in local coordinates $\left(\tilde{u}_q', \tilde{v}_q'\right)$



Let $q = n$: After a change of coordinates $u' = \tilde{u}_n, v' = \tilde{v}_n - \frac{1}{\mu(\tilde{u}_n)}$ and n blow-ups in $(0,0)$, $(g \circ e)$ is good for every point of D_Z in local coordinates (u'_s, v'_s) and holomorphic for every point of D_Z in local coordinates $(\tilde{u}'_n, \tilde{v}'_n)$. As before the intersection points P_i of the strict transform of $S_i : \mu_i\,(t)\,y = t^{q_i}$ wit D_Z is given by

$$P_i = \left(0, \frac{1}{\mu_i\,(0)}\right)$$

in the suitable coordinates. Now let $q_i = q$ and $\mu_i\,(0) = \mu\,(0)$, i. e. we consider $S : \mu\,(t)\,y = t^q$ (Notice, that S corresponds to our given ψ!)

- $q < n$, i.e. $k := q + 1 \neq n$: $\overline{S} : u_k^{k-1} v_k^{k-1} \left(1 - \mu_i\,(u_k v_k)\,v_k\right) = 0$ and by coordinate transform we have: $\overline{S} : u_k^{k-1} v_k^{k-1} v' = 0$. As $v' = \tilde{u}'_q \tilde{v}'_q$ we get the unique intersection point $P = (0,0)$.
- $q = n$: In the same way we get the intersection point $P = (0,0)$.

$\square$

2.1.2 Topology of $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$

D'_Z is a normal crossing divisor, so locally at a crossing point D'_Z has the form $\{uv = 0\}$ and at a smooth point D'_Z has the form $\{u = 0\}$. Thus we can describe $\partial \widetilde{Z}\,(D'_Z)$ in local coordinates:

- real blow up with respect to $\{u = 0\}$: $\zeta = (0, \theta_u, |v|\,, \theta_v)$ (with $v = |v| \cdot e^{i\theta_v}$)
- real blow up with respect to $\{uv = 0\}$: $\zeta = (0, \theta_u, 0, \theta_v)$

Now we take the fiber of $\vartheta \in \mathbb{S}^1$, i. e. we consider $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$. For every fixed $|v|$ we have a bijection $\{(0, \theta_u, |v|\,, \theta_v)\} \leftrightarrow \mathbb{S}^1$, thus, following the nomenclature of C. Sabbah, we can interpret $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ as a system of pipes, which furthermore is homeomorphic to a disc ([Sab13], p. 203)
Note, that for all i the strict transform of the divisor component S_i intersects the real blow up divisor $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ in a unique point P_i.

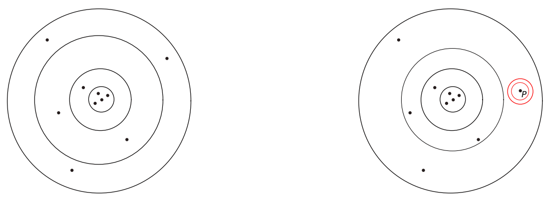
Remark 2.12:

1. Also the irreducible components $\widetilde{S}_j$ intersect $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ in distinct points $\widetilde{P}_j$. This follows directly from Assumption 1.2.

2. According to Assumption 1.1 we have ($q_i \neq q_j$ or $\mu_i(0) \neq \mu_j(0)$ for $i \neq j$). This induces $P_i \neq P_j$.

3. Every intersection point $P_i, \widetilde{P}_j$ may be interpreted as a 'leak' in the system of pipes $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ (see [Sab13], p. 204). Thus topologically we can think of $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ as a disc with singularities, which come from the intersection with $\widetilde{S}_j$ and $\overline{S_i}$.

$\psi = 0$ $\qquad\qquad$ $\psi \neq 0$



2.1.3 Explicit Description of B_ψ^ϑ

Remember the definition of $B_\psi^\vartheta := \left\{\zeta \in \left(\widetilde{p \circ e}\right)^{-1}(\vartheta) \mid \left(\mathcal{H}^\bullet \mathcal{F}_\psi\right)_\zeta \neq 0\right\}$. Obviously we have:

$$\zeta \in B_\psi^\vartheta \Leftrightarrow \left(\mathcal{H}^\bullet \mathcal{F}_\psi\right)_\zeta \neq 0 \Leftrightarrow e^{(g \circ e)} \in \mathcal{A}_\vartheta^{mod\, D'_Z} \text{ near } \zeta$$

(The second equivalence follows by considering $\mathrm{DR}^{mod\, \overline{D}_Z}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e}\right)$, whereof we know that it has cohomology in degree 0 at most (cf. Proposition 2.8 and Lemma 3.2).) Thus we need to take a closer look at the exponent $g \circ e$. Therefore we use the following Lemma (cf. [Sab13], 9.4):

Lemma 2.13: *Let u, v local coordinates of the divisor D_Z, such that $f(u,v)$ holomorphic or good, i. e.*

$$f(u,v) = \frac{1}{u^m v^n}\beta(u,v), \quad \textit{whereby } \beta \textit{ holomorphic and } \beta(0,v) \neq 0.$$

Then $e^{f(u,v)} \in \mathcal{A}_\vartheta^{mod\, D_Z}$ around a given point $\zeta \in \left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ if and only if

$$f \textit{ holomorphic in } \zeta \textit{ or } arg\left(\beta(0,v)\right) - m\theta_u - n\theta_v \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \textit{ mod } 2\pi.$$

Lemma 2.14:

1. *Let $S_i : \mu_i(t)\, y = t^{q_i}$, $\psi_i(t) = \mu_i(t)\, t^{-q_i}$ with $\mu_i(0) \neq 0$ and P_i the corresponding intersection point. Then we have: $P_i \in B_\psi^\vartheta \Leftrightarrow \psi_i \leq_\vartheta \psi$.*

2. *Let $\widetilde{P}_j$ the intersection points of $\widetilde{S}_j$ with $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$. Then we have: $\left\{\widetilde{P}_j \mid j = 1, \ldots, J\right\} \subset B_\psi^\vartheta \Leftrightarrow 0 \leq_\vartheta \psi$.*

Proof: This follows by examining goodness condition of $g \circ e$ near the intersection points and by Remark 2.4 $\square$

Definition 2.15: For abbreviation we define the following sets:

- $\mathcal{P}^{\vartheta} := \{\widetilde{P}_j \mid j = 1 \ldots, J\} \cup \{P_i \mid i = 1, \ldots I\} \subset (p \circ e)^{-1}(\vartheta)$
- $\mathcal{P}^{\vartheta}_{\psi} := \mathcal{P}^{\vartheta} \cap B^{\vartheta}_{\psi}$

Lemma 2.16: *Let $\psi = 0$. Then the fundamental group $\pi_1\left(B^{\vartheta}_0 \setminus \mathcal{P}^{\vartheta}_0\right)$ is a free group of rank $\#\left(\mathcal{P}^{\vartheta}_0\right)$.*

Proof: This follows from the fact, that B^{ϑ}_0 emerges from glueing the following sets of points:

- $M_1 = \{\zeta = (0, \theta_{u_1}, |v_1|, \theta_{v_1}) \mid v_1 \neq 0\}$
- $M_2 = \{\zeta = (0, \theta_{u_1}, 0, \theta_{v_1}) \mid \theta_{v_1} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\}$
- $M_3 = \{\zeta = (0, \theta_{u_k}, |v_k|, \theta_{v_k}) \mid \theta_{v_k} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\}$
- $M_4 = \{\zeta = (0, \theta_{\tilde{u}_n}, |\tilde{v}_n|, \theta_{\tilde{v}_n}) \mid \theta_{\tilde{v}_n} \in \left(\frac{\pi}{2} - n\vartheta, \frac{3\pi}{2} - n\vartheta\right)\}$

Since M_1 contains the $\widetilde{P}_j$'s and M_2, M_3, M_4 are simply connected and contain the relevant P_i's, this shows the claim. $\square$

Lemma 2.17: *Let $\psi \neq 0$, i. e. ψ is given by $\psi(t) = \mu(t)\, t^{-q}$. Then $\pi_1\left(B^{\vartheta}_{\psi} \setminus \mathcal{P}^{\vartheta}_{\psi}\right)$ is a free group of rank $\#\left(\mathcal{P}^{\vartheta}_{\psi}\right)$.*

Proof: Explicitely we have to prove:

1. For $\vartheta \in \left(\frac{\frac{\pi}{2} + arg(-\mu(0))}{q}, \frac{\frac{3\pi}{2} + arg(-\mu(0))}{q}\right)$ $mod\ \frac{2\pi}{q}$: $\pi_1\left(B^{\vartheta}_{\psi} \setminus \mathcal{P}^{\vartheta}_{\psi}\right)$ is a free group of rank $\#\{\widetilde{P}_j\} + \#\{P_i \mid P_i \in B^{\vartheta}_{\psi}\}$
2. For $\vartheta \notin \left(\frac{\frac{\pi}{2} + arg(-\mu(0))}{q}, \frac{\frac{3\pi}{2} + arg(-\mu(0))}{q}\right)$ $mod\ \frac{2\pi}{q}$: $\pi_1\left(B^{\vartheta}_{\psi} \setminus \mathcal{P}^{\vartheta}_{\psi}\right)$ is a free group of rank $\#\{P_i \mid P_i \in B^{\vartheta}_{\psi}\}$

For $k \leq q$ we have: $\zeta = (0, \theta_{u_k}, |v_k|, \theta_{v_k}) \in B^{\vartheta}_{\psi} \Leftrightarrow arg(-\mu(0)) - q\vartheta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. This explains the sub-division into the two cases above. As in the previous lemma B^{ϑ}_{ψ} consists of glueing simply connected sets (additionally taking the sets of the branching part into account). $\square$

Descriptively, this means, that there are no other holes' in B_ψ^ϑ than the singularities P_i, which arise from the intersection with S_i and possibly – depending on the choice of ϑ if $\psi \neq 0$ – the singularities $\widetilde{P}_j$, which arise from the intersection with $\widetilde{S}_j$.
Remark, that an open interval of the boundary of B_ψ^ϑ lies in B_ψ^ϑ! This holds because for $k = n$ we have

$$(0, \vartheta, 0, \theta_{\tilde{v}_n}) \in B_\psi^\vartheta \Leftrightarrow \theta_{\tilde{v}_n} \in \left(\frac{\pi}{2} - n\vartheta, \frac{3\pi}{2} - n\vartheta\right)$$

both if $q < n$ and if $q = n$.

2.1.4 Dimension of $\mathbb{H}_c^1\left(B_\psi^\vartheta, \mathcal{F}_\psi\right)$

Proposition 2.18: *On $B_\psi^\vartheta \setminus \mathcal{P}_\psi^\vartheta$ the perverse sheaf $\mathcal{F}_\psi$ has cohomology in degree 0 at most.*

Proof: We know that $e^+\mathcal{M}$ has regular singularities along the divisor and that $\mathcal{E}^{g(t,y)}$ is good or even holomorphic. Moreover the divisor is normal crossing except possibly at the intersection points P_i, $\widetilde{P}_j$. Thus we can apply [Sab13], Corollary 11.22. □

Lemma 2.19 ([Sab], Prop 1.1.6)**:** *Let Δ be an open disc and Δ' an open subset of the closure $\overline{\Delta}$, consisting of Δ and a connected open subset of $\partial\overline{\Delta}$. Let C be a finite set of points in Δ. Let $\mathcal{M}$ be a regular singular $\mathcal{D}_{\Delta'}$-module and consider a perverse sheaf $\mathcal{F}$ with singularities in $c \in C$ only. Then $\mathbb{H}_c^k(\Delta', \mathcal{F}) = 0$ for $k \neq 1$ and* $\dim$ *$\mathbb{H}_c^1(\Delta', \mathcal{F})$ is equal to the sum of the dimensions of the vanishing cycle spaces at $c \in C$.*

Proof: For a proof we refer to [Sab]. □

As a direct consequence we see:

Theorem 2.20: *For $\vartheta \in \mathbb{S}^1$ we have* $\dim\left(\mathcal{L}_{\leq\psi}\right)_\vartheta = \dim\left(\widetilde{\mathcal{L}}_{\leq\psi}\right)_\vartheta$.

Proof: This follows from Corollary 2.5 and Lemma 2.19. □

Remark 2.21: In an analogous way we can determine $\mathcal{L}$ on $\mathbb{S}_1$. First let us recall that $\mathcal{L} = j_*\mathcal{L}'$, whereby $\mathcal{L}'$ denotes the local system associated to $\mathcal{N} := \mathcal{H}^0 p_+\left(\mathcal{M} \otimes \mathcal{E}^{\frac{1}{y}}\right)$. Consider a small circle $\mathbb{S}_\epsilon^1$ around $0 \in \Delta$. Then we have

$$\mathcal{L}_{|S^1} \cong \mathcal{L}'_{|S_\epsilon^1}$$

On $\mathbb{S}_\epsilon^1$ we can identify $\mathcal{L}' = \mathcal{H}^0 \operatorname{DR}_{\widetilde{\Delta}}^{mod\, 0}(\mathcal{N})$ (since the growing condition *mod* 0 is irrelevant

outside of $\partial\widetilde{\Delta}$). Furthermore in this situation, the isomorphism

$$\Omega : \mathcal{H}^0\, \mathrm{DR}^{mod\,0}_{\widetilde{\Delta}}(\mathcal{N})_{|\mathbb{S}^1_\epsilon} \xrightarrow{\cong} \mathcal{H}^1 R\widetilde{p}_*\, \mathrm{DR}^{mod\,D}\left(\mathcal{M}\otimes\mathcal{E}^{\frac{1}{y}}\right)_{|S^1_\epsilon}$$

holds (analogously to Theorem 2.6). Now as before we can describe the right hand side in topological terms. Since $\mathcal{E}^{\frac{1}{y}}$ is a good (or even holomorphic) connection, we know that $\mathrm{DR}^{mod\,D}\left(\mathcal{M}\otimes\mathcal{E}^{\frac{1}{y}}\right)$ has cohomology in degree zero at most and thus corresponds to a sheaf. By taking the stalk at a point $\rho\in\mathbb{S}^1_\epsilon$ we get

$$\left(\mathcal{H}^0\, \mathrm{DR}^{mod\,0}_{\widetilde{\Delta}}(\mathcal{N})\right)_\rho \cong H^1_c\left(\widetilde{p}^{-1}(\rho), \mathrm{DR}^{mod\,D}\left(\mathcal{M}\otimes\mathcal{E}^{\frac{1}{y}}\right)\right).$$

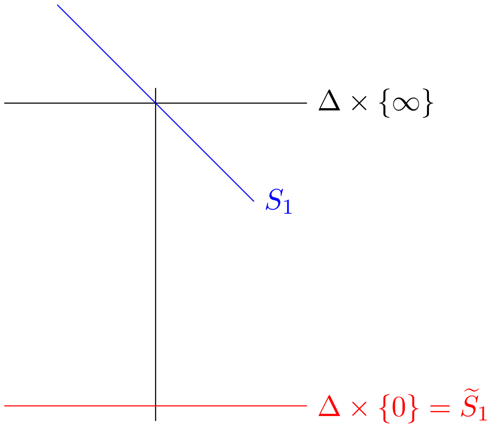
$\widetilde{p}^{-1}(\rho)$ corresponds to the projective line with a real blow up in the point ∞ (denoted by $\mathbb{S}^1_\infty$). We can interpret this space as a disc with boundary $\mathbb{S}^1_\infty$. Furthermore by choosing ϵ small enough, we know that every irreducible component S_i resp. $\widetilde{S}_j$ of $SS(\mathcal{M})$ meets the (inner of the) disc in exactly one point (P_i resp. $\widetilde{P}_j$).
The sheaf is supported everywhere away from the boundary, on the boundary its supported on an open hemicycle of $\mathbb{S}^1_\infty$ (i.e. the points where it fulfills the moderate growth condition). Thus the dimension of $H^1_c\left(\widetilde{p}^{-1}(\rho), \mathrm{DR}^{mod\,D}\left(\mathcal{M}\otimes\mathcal{E}^{\frac{1}{y}}\right)\right)$ depends on the dimension of the vanishing cycle spaces in the points P_i and $\widetilde{P}_j$:

$$\dim\, H^1_c\left(\widetilde{p}^{-1}(\rho), \mathrm{DR}^{mod\,D}\left(\mathcal{M}\otimes\mathcal{E}^{\frac{1}{y}}\right)\right) = \dim\sum_{i\in I}\Phi_{P_i} + \sum_{j\in J}\Phi_{\widetilde{P}_j}.$$

We will use this topological description in the following chapter, supply it to the explicit example an determine a set of linear Stokes data.

3 Proof of Theorem 1.5: Explicit example for the determination of Stokes data

As already mentioned in the introduction, we will consider the following explicit example:
Let $X = \Delta\times\mathbb{P}^1$ and $\mathcal{M}$ a meromorphic connection on X of rank r with regular singularities along its divisor. Let the singular locus $SS(\mathcal{M})$ be of the following form: $SS(\mathcal{M}) = \{t\cdot y\cdot(t-y)\cdot x = 0\}$. Denote the irreducible components $(S_1 : y = t)$ and $(\widetilde{S}_1 : x = 0)$.



The following chapter is devoted to prove Theorem 1.5 and thereby determine a set of Stokes data to $\mathcal{H}^0 p_+ \left(\mathcal{M} \otimes \mathcal{E}^{\frac{1}{y}}\right)$.

3.1 Stokes-filtered local system

According to Theorem 2.1, $\hat{\mathcal{N}} = \mathcal{H}^0 p_+ \left(\mathcal{M} \otimes \mathcal{E}^q\right)_0^\wedge$ can be decomposed to:

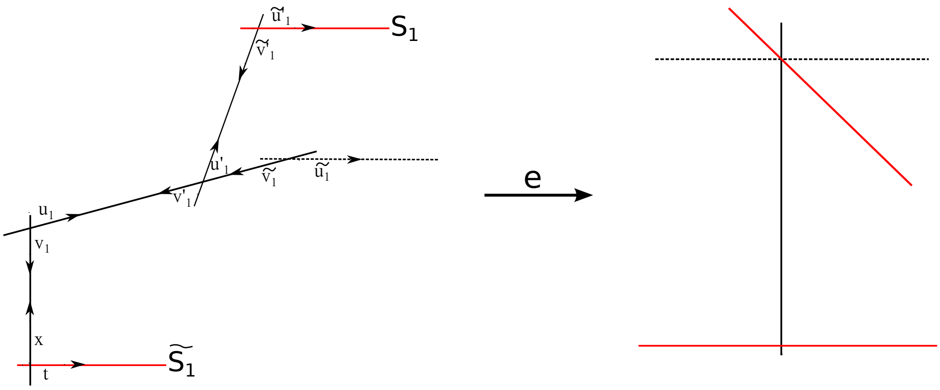
$$\hat{\mathcal{N}} \cong R_0 \oplus \left(R_1 \otimes \mathcal{E}^{\frac{1}{t}}\right)$$

whereby $rk\left(R_0\right) = rk\left(R_1\right) = r$. Now we will describe the Stokes structure via the isomorphism Ω (Theorem 2.6). Ω identifies $\left(\mathcal{L}_{\leq \psi}\right)_\vartheta \cong H^1\left(\left(\widetilde{p \circ e}\right)^{-1}(\vartheta), \beta_{\psi,!}^{\vartheta} \mathcal{F}_\psi\right)$.

Lemma 3.1: *There exists a sequence of point blow ups $e : Z \to X$, such that*

1. *the singular support of $\mathcal{M}$ becomes a normal crossing divisor*
2. *for both $\psi = 0$ and $\psi = \frac{1}{t}$ the exponent $g \circ e$ is holomorphic or good in every point.*

Proof: The following blow up of the divisor satisfies the requested conditions:



(i. e. a point-blow-up in $(t, y) = 0$ and a second point-blow-up in $(\tilde{u}_1, \tilde{v}_1) = (0, 1)$) $\square$

We will denote the resulting normal crossing divisor by $\overline{D}_Z$. The fiber over ϑ with respect to the blow up along $\overline{D}_Z$ is homeomorphic to a closed disc with two 'holes'. We will denote it by $\overline{A} \times \{\vartheta\}$.

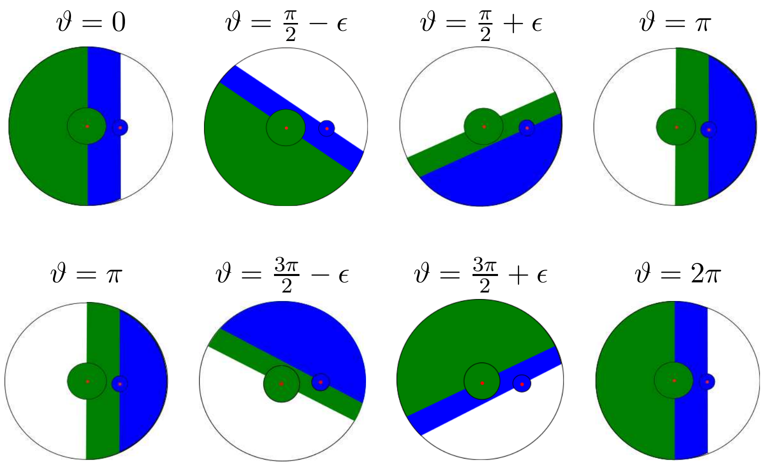
Lemma 3.2: $\mathrm{DR}^{mod\,\overline{D}_Z}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g\circ e}\right)$ *has cohomology in degree* 0 *at most.*

Proof: Since $\overline{D}_Z$ is normal crossing and $g \circ e$ is good or holomorphic (Lemma 3.1) the claim follows by [Sab13], Prop. 8.15 and Cor. 11.22. $\square$

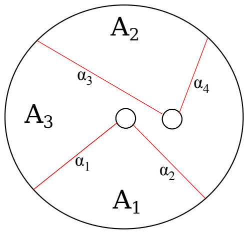
Consider the map $\kappa : \widetilde{Z}\left(\overline{D}_Z\right) \to \widetilde{Z}\left(D_Z\right)$. Restricting it to a fiber $\kappa_\vartheta : \overline{A} \times \{\vartheta\} \to \left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$, it is just the identity except at the points $\widetilde{P}_1$ and P_1, where it describes the "collapse" of the real blow ups of $\widetilde{P}_1$ and P_1 back to these points. $\mathrm{DR}^{mod\,\overline{D}_Z}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g\circ e}\right)$ has cohomology in degree 0 at most and therefore corresponds to a local system $\mathcal{G}_\psi$. Then $\mathrm{DR}^{mod\,D_Z}\left(e^+\mathcal{M} \otimes \mathcal{E}^{g\circ e}\right)$ corresponds to the perverse sheaf $\mathcal{F}_\psi$ given by $0 \to \kappa_*\mathcal{G}_\psi \to 0 \to 0$. Let $\mathcal{G}_\psi^\vartheta$ the restriction to the support $\kappa^{-1}\left(B_\psi^\vartheta\right)$ in the fiber $\overline{A} \times \{\vartheta\}$ and $\overline{\beta}_\psi^\vartheta : \kappa^{-1}\left(B_\psi^\vartheta\right) \hookrightarrow \overline{A} \times \{\vartheta\}$ the open inclusion. Because of Proposition 2.8 we have:

$$H^1\left(\overline{A} \times \{\vartheta\}, \overline{\beta}_{\psi,!}^\vartheta \mathcal{G}_\psi^\vartheta\right) \cong H^1\left(\left(\widetilde{p \circ e}\right)^{-1}(\vartheta), \beta_{\psi,!}^\vartheta \mathcal{F}_\psi^\vartheta\right)$$

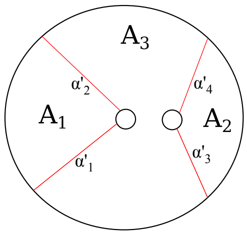
Furthermore we denote by $\mathcal{K}$ the local system on $\widetilde{Z}\left(\overline{D}_Z\right)$ corresponding to the pullback connection $e^+\mathcal{M}$ of rank $rk(\mathcal{M})$ (see Cor. 8.3 in [Sab13]) and by $\mathcal{K}^\vartheta$ its restriction to $\overline{A} \times \{\vartheta\}$. Then obviously $\overline{\beta}_{\psi,!}^\vartheta \mathcal{G}_\psi^\vartheta$ equals $\overline{\beta}_{\psi,!}^\vartheta \overline{\beta}_\psi^{\vartheta\,-1} \mathcal{K}^\vartheta$ for all ψ. For brevity we will write $\overline{\beta}_{\psi,!}^\vartheta \mathcal{K}^\vartheta$ instead of $\overline{\beta}_{\psi,!}^\vartheta \overline{\beta}_\psi^{\vartheta\,-1} \mathcal{K}^\vartheta$ in the following. Thus it is enough to examine $\left(\mathcal{L}_{\leq\psi}\right)_\vartheta \cong H^1\left(\overline{A} \times \{\vartheta\}, \overline{\beta}_{\psi,!}^\vartheta \mathcal{K}^\vartheta\right)$. With the blow up e constructed in the proof of Lemma 3.1 we can determine the open subset $B_\vartheta^\psi \subset \left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ and we receive the following pictures, which show $\left(\widetilde{p \circ e}\right)^{-1}(\vartheta)$ (resp. $\overline{A} \times \{\vartheta\}$) and the subsets B_0^ϑ, $B_{\frac{1}{t}}^\vartheta$. One can see very clearly that by passing the Stokes directions $\pm\frac{\pi}{2}$, the relation of the subsets B_0^ϑ and $B_{\frac{1}{t}}^\vartheta$ changes from $B_0^\vartheta \subset B_{\frac{1}{t}}^\vartheta$ to $B_{\frac{1}{t}}^\vartheta \subset B_0^\vartheta$ and vice versa.



In this situation we can compute the first cohomology groups $H^1\left(\overline{A}\times\{\vartheta\},\overline{\beta}^{\vartheta}_{\psi,!}\mathcal{K}^{\vartheta}\right)$ in another way, namely by using Čech cohomology.
Therefore consider the following curves $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ in $\overline{A}$:



Then for $\vartheta_0\in[0,\frac{\pi}{2})$ the closed covering $\mathfrak{A}=A_1\cup A_2\cup A_3$ of $\overline{A}$ defines a Leray covering of $\beta^{\vartheta_0}_{\psi,!}\mathcal{K}^{\vartheta_0}$ and for $\beta^{\vartheta_1}_{\psi,!}\mathcal{K}^{\vartheta_1}$, whereby $\vartheta_1:=\vartheta_0+\pi$. This can be proved easily by using the exact sequence of Chapter 3.3.4 and doing the same calculations for the restriction to the intersections α_k.
For $\vartheta_0\in[\frac{\pi}{2},\pi)$ the following curves define a Leray covering for $\beta^{\vartheta_0}_{\psi,!}\mathcal{K}^{\vartheta_0}$ and $\beta^{\vartheta_1}_{\psi,!}\mathcal{K}^{\vartheta_1}$:



Thus we can conclude:

Lemma 3.3: *For every pair of angles $(\vartheta_0,\vartheta_1:=\vartheta_0+\pi)$ of $\mathbb{S}^1$ the construction above defines a closed covering $\mathfrak{A}$ of $\overline{A}$, such that $\mathfrak{A}$ is a common Leray covering of $\beta^{\vartheta_i}_{\psi,!}\mathcal{K}^{\vartheta_i}$ ($i=1,2$ and $\psi=0,\frac{1}{t}$). Consequently we get an isomorphism*

$$H^1\left(\overline{A}\times\{\vartheta_i\},\overline{\beta}^{\vartheta_i}_{\psi,!}\mathcal{K}^{\vartheta_i}\right)\to\check{H}^1\left(\mathfrak{A},\overline{\beta}^{\vartheta_i}_{\psi,!}\mathcal{K}^{\vartheta_i}\right)$$

In the following lemma we will compute the cohomology groups concretely for $\vartheta=0$ and $\vartheta=\pi$.

Lemma 3.4:

- $H^1\left(\overline{A}\times\{0\},\overline{\beta}^{0}_{0,!}\mathcal{K}^{0}\right)\cong\mathcal{K}^0_{x_1}$
- $H^1\left(\overline{A}\times\{0\},\overline{\beta}^{\frac{1}{t}}_{0,!}\mathcal{K}^{0}\right)=H^1\left(\overline{A}\times\{0\},\overline{\beta}^{0}_{!}\mathcal{K}^{0}\right)\cong\mathcal{K}^0_{x_1}\oplus\mathcal{K}^0_{x_3}$
- $H^1\left(\overline{A}\times\{\pi\},\overline{\beta}^{\frac{1}{t}}_{\pi,!}\mathcal{K}^{\pi}\right)\cong\mathcal{K}^\pi_{x_4}$
- $H^1\left(\overline{A}\times\{\pi\},\overline{\beta}^{0}_{\pi,!}\mathcal{K}^{\pi}\right)=H^1\left(\overline{A}\times\{\pi\},\overline{\beta}^{\pi}_{!}\mathcal{K}^{\pi}\right)\cong\mathcal{K}^\pi_{x_2}\oplus\mathcal{K}^\pi_{x_4}$

Proof: Just notice that $\check{C}^0=0$ and thus $\check{H}^1=\check{C}^1$. ☐

3.2 Stokes data associated to $\mathcal{L}$

Using the functor constructed in [HS11] we can associate a set of Stokes data to the Stokes-filtered local system $\mathcal{L}$ described above.

Construction 3.5: Fix two intervals

$$I_0 = (0 - \epsilon, \pi + \epsilon)\,, \;\; I_1 = (-\pi - \epsilon, 0 + \epsilon)$$

of length $\pi + 2\epsilon$ on $\mathbb{S}^1$, such that the intersection $I_0 \cap I_1$ consists of $(0 - \epsilon, 0 + \epsilon)$ and $(\pi - \epsilon, \pi + \epsilon)$. Observe, that the intersections do not contain the Stokes directions $-\frac{\pi}{2}, \frac{\pi}{2}$. To our given local system $\mathcal{L}$ we associate:

- a vector space associated to the angle 0, i. e. the stalk $\mathcal{L}_0$. It comes equipped with the Stokes filtration.
- a vector space associated to the angle π, i. e. the stalk $\mathcal{L}_\pi$, coming equipped with the Stokes filtration.
- vector spaces associated to the intervals I_0 and I_1, i. e. the global sections $\Gamma(I_0, \mathcal{L})$, $\Gamma(I_1, \mathcal{L})$
- a diagram of isomorphisms (given by the natural restriction to the stalks):

$$\begin{tikzcd}
& \Gamma(I_0, \mathcal{L}) \arrow[dl, "a_0"'] \arrow[dr, "a_0'"] & \\
\mathcal{L}_0 & & \mathcal{L}_\pi \\
& \Gamma(I_1, \mathcal{L}) \arrow[ul, "a_1"] \arrow[ur, "a_1'"'] &
\end{tikzcd}$$

The filtrations on the stalks are opposite with respect to the maps $a_0' a_0^{-1}$ and $a_1 {a'}_1^{-1}$, i. e.

$$\mathcal{L}_0 = \bigoplus_{\phi \in \left\{0, \frac{1}{t}\right\}} L_{\leq \phi, 0} \cap a_0' a_0^{-1}\left(L_{\leq \phi, \pi}\right), \qquad \mathcal{L}_\pi = \bigoplus_{\phi \in \left\{0, \frac{1}{t}\right\}} L_{\leq \phi, \pi} \cap a_1 {a'}_1^{-1}\left(L_{\leq \phi, 0}\right)$$

Furthermore we know, by using the isomorphisms Ω and Γ, that $\mathcal{L}_0 \cong \mathcal{K}_{x_1} \oplus \mathcal{K}_{x_3}$ whereby $\mathcal{L}_{\leq 0,0} \cong \mathcal{K}_{x_1}$. Thus we have a splitting $\mathcal{L}_0 = G_0 \oplus G_{\frac{1}{t}}$ with $G_0 = \mathcal{L}_{\leq 0,0}$. The same holds for $\mathcal{L}_\pi$: $\mathcal{L}_\pi = H_0 \oplus H_{\frac{1}{t}}$ with $H_{\frac{1}{t}} = \mathcal{L}_{\leq \frac{1}{t}, \pi}$.

Thus the set of data $\left(G_0, G_{\frac{1}{t}}, H_0, H_{\frac{1}{t}}, S_0^\pi, S_\pi^0\right)$ with:

- $\mathcal{L}_0 = G_0 \oplus G_{\frac{1}{t}}$ and $\mathcal{L}_\pi = H_{\frac{1}{t}} \oplus H_0$
- $S_0^\pi : G_0 \oplus G_{\frac{1}{t}} \xrightarrow{a_0^{-1}} \Gamma(I_0, \mathcal{L}) \xrightarrow{a_0'} H_0 \oplus H_{\frac{1}{t}}$

- $S_{\pi}^{0}: H_{\frac{1}{t}} \oplus H_0 \xrightarrow{a_1^{-1}} \Gamma\left(I_1, \mathcal{L}\right) \xrightarrow{a_1'} G_{\frac{1}{t}} \oplus G_0$

describes a set of Stokes data associated to the local system $\mathcal{L}$.
By exhaustivity of the filtrations on $\mathcal{L}_{\vartheta}$ ($\vartheta = 0, \pi$) the isomorphism Ω induces

$$\mathcal{L}_{\vartheta} \cong H^1\left(\overline{A} \times \vartheta, \overline{\beta}_!^{\vartheta} \mathcal{K}^{\vartheta}\right)$$

where $\overline{\beta}^{\vartheta} : B^{\vartheta} \hookrightarrow \overline{A} \times \vartheta$ corresponds to the $\overline{\beta}_{\psi}^{\vartheta}$ with $\psi \geq_{\vartheta} \phi$ for $\psi, \phi \in \Phi = \{0, \frac{1}{t}\}$ and $B^{\vartheta} := B_{\psi}^{\vartheta} \supset B_{\phi}^{\vartheta}$.
In the same way, for an open interval $I \subset \mathbb{S}^1$ let $\mathcal{K}^I$ be the restriction of $\mathcal{K}$ to $\overline{A} \times I$ and define $\overline{\beta}^I : B^I \hookrightarrow \overline{A} \times I$ the inclusion of the subspace B^I, which is the support of $\mathrm{DR}^{mod\, D}(\mathcal{M} \otimes \mathcal{E}^{\frac{1}{y}})$ according to Remark 2.21. Notice that $\overline{\beta}^I_{|\overline{A} \times \vartheta} = \overline{\beta}^{\vartheta}$. Then, by the isomorphism Ω, we identify

$$\Gamma\left(I, \mathcal{L}\right) \cong H^1\left(\overline{A} \times I, \overline{\beta}_!^{I} \mathcal{K}^{I}\right).$$

With these isomorphisms we have restriction morphisms (according to the restrictions to the stalks)

$$\rho_{\vartheta} : H^1\left(\overline{A} \times I_0, \overline{\beta}_!^{I_0} \mathcal{K}^{I_0}\right) \xrightarrow{\cong} H^1\left(\overline{A} \times \vartheta, \overline{\beta}_!^{\vartheta} \mathcal{K}^{\vartheta}\right) \text{ for } \vartheta \in I_0$$
$$\rho'_{\vartheta} : H^1\left(\overline{A} \times I_1, \overline{\beta}_!^{I_1} \mathcal{K}^{I_1}\right) \xrightarrow{\cong} H^1\left(\overline{A} \times \vartheta, \overline{\beta}_!^{\vartheta} \mathcal{K}^{\vartheta}\right) \text{ for } \vartheta \in I_1$$

This yields to a new way of describing Stokes data associated to the local system $\mathcal{L}$:

Theorem 3.6: *Fix the following data:*

- *vector spaces* $L_0 := H^1\left(\overline{A} \times \{0\}, \overline{\beta}_!^{0} \mathcal{K}^{0}\right)$ *and* $L_1 := H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}_!^{\pi} \mathcal{K}^{\pi}\right)$
- *morphisms* $\sigma_0^{\pi} := \rho_{\pi} \circ \rho_0^{-1}$ *and* $\sigma_{\pi}^{0} := \rho'_0 \circ {\rho'}_{\pi}^{-1}$ *, with* ρ_{ϑ} *and* ρ'_{ϑ} *defined as above.*

Then

$$\left(L_0, L_1, \sigma_0^{\pi}, \sigma_{\pi}^{0}\right)$$

defines a set of Stokes data for $\mathcal{H}^0 p_+ \left(\mathcal{M} \otimes \mathcal{E}^{\frac{1}{y}}\right)$.

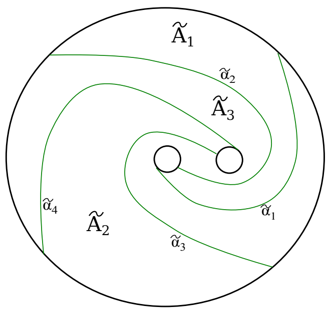
Proof: The isomorphism $\Omega : \mathcal{L}_{\vartheta} \to H^1\left(\overline{A} \times \{\vartheta\}, \overline{\beta}_!^{\vartheta} \mathcal{K}^{\vartheta}\right)$ passes the filtrations on $\mathcal{L}_0$ and $\mathcal{L}_{\pi}$ to L_0 and L_1 and therefore yields to a suitable graduation of $L_0 = \widetilde{G}_0 \oplus \widetilde{G}_{\frac{1}{t}} \cong \mathcal{K}_{x_1}^0 \oplus \mathcal{K}_{x_3}^0$ and $L_1 = \widetilde{H}_{\frac{1}{t}} \oplus \widetilde{H}_0 \cong \mathcal{K}_{x_4}^{\pi} \oplus \mathcal{K}_{x_2}^{\pi}$.
On the level of graduated spaces $\Omega : G_0 \oplus G_{\frac{1}{t}} \to \widetilde{G}_0 \oplus \widetilde{G}_{\frac{1}{t}}$ (resp. $H_{\frac{1}{t}} \oplus H_0 \to \widetilde{H}_{\frac{1}{t}} \oplus \widetilde{H}_0$) is obviously described by a block diagonal matrix. Furthermore, by definition of ρ and ρ', the maps σ_0^{π} and σ_{π}^{0} can be read as $\sigma_0^{\pi} = \Omega \circ S_0^{\pi} \circ \Omega^{-1}$ and $\sigma_{\pi}^{0} = \Omega \circ S_{\pi}^{0} \circ \Omega^{-1}$. Since, according to the construction in 3.5, S_0^{π} is upper block triangular (resp. S_{π}^{0} is lower block triangular) the same holds for σ_0^{π} (resp. σ_{π}^{0}). $\square$

3.3 Explicit computation of the Stokes matrices

The determination of the Stokes data thus corresponds to the following picture:

$$\begin{tikzcd}
 & H^1\left(\overline{A}\times I_0, \overline{\beta}_!^{I_0}\mathcal{K}\right) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
H^1\left(\overline{A}\times\{0\}, \overline{\beta}_!^{0}\mathcal{K}^0\right) \arrow[rr, bend left=10] & & H^1\left(\overline{A}\times\{\pi\}, \overline{\beta}_!^{\pi}\mathcal{K}^\pi\right) \arrow[ll, bend left=10] \\
 & H^1\left(\overline{A}\times I_1, \overline{\beta}_!^{I_1}\mathcal{K}\right) \arrow[ul, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

In Lemma 3.4 we have already computed the cohomology groups for $\vartheta = 0$ and $\vartheta = \pi$ using the Leray covering $\mathfrak{A}$. Now for $l = 0, 1$ we fix a diffeomorphism $\overline{A}\times I_l \xrightarrow{\sim} \overline{A}\times I_l$ by lifting the vector field ∂_ϑ to $\overline{A}\times\mathbb{S}^1$ such that the lift is equal to ∂_ϑ away from a small neighborhood of $\partial\overline{A}$ and such that the diffeomorphism induces $B^{I_l} \xrightarrow{\cong} B^{\vartheta_{l+1}}\times I_l$ where $\vartheta_1 = \pi$ and $\vartheta_2 = 0$. It induces a diffeomorphism $\overline{A}\times\{\vartheta_l\} \xrightarrow{\cong} \overline{A}\times\{\vartheta_{l+1}\}$ and an isomorphism between the push forward of $\mathcal{K}^{\vartheta_l}$ and $\mathcal{K}^{\vartheta_{l+1}}$. Moreover it sends the boundary ∂B^{ϑ_l} to $\partial B^{\vartheta_{l+1}}$ (i. e. the boundaries are rotated in counter clockwise direction by the angle π). Via this diffeomorphism the curves $\alpha_i \subset \overline{A}\times\vartheta_l$ are sent to curves $\widetilde{\alpha}_i \subset \overline{A}\times\vartheta_{l+1}$ and therefore induce another Leray covering $\widetilde{\mathfrak{A}}$ of $H^1\left(\overline{A}\times\{\vartheta_{l+1}\}, \overline{\beta}_!^{\vartheta_{l+1}}\mathcal{K}^{\vartheta_{l+1}}\right)$.



Explicitly, for $l = 0$ we get a Leray covering $\widetilde{\mathfrak{A}}$ of $H^1\left(\overline{A}\times\{\pi\}, \overline{\beta}_!^{\pi}\mathcal{K}^\pi\right)$ and, as in the previous chapter, we get an isomorphism $\widetilde{\Gamma}_\pi : H^1\left(\overline{A}\times\{\pi\}, \overline{\beta}_!^{\pi}\mathcal{K}^\pi\right) \to \check{H}^1\left(\widetilde{\mathfrak{A}}, \overline{\beta}_!^{\pi}\mathcal{K}^\pi\right)$, which gives us:

$$H^1\left(\overline{A}\times\{\pi\}, \overline{\beta}_!^{\pi}\mathcal{K}^\pi\right) \cong \mathcal{K}^\pi\left(\widetilde{\alpha}_1\right) \oplus \mathcal{K}^\pi\left(\widetilde{\alpha}_3\right) \cong \mathcal{K}^\pi_{x_1} \oplus \mathcal{K}^\pi_{x_3}$$

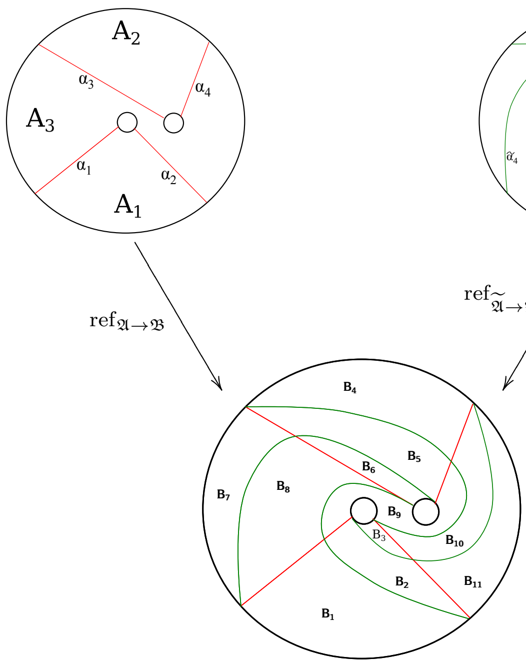
Thus the above diffeomorphism leads to an isomorphism

$$\mu_0^\pi : \mathcal{K}_{x_1}^0 \oplus \mathcal{K}_{x_3}^0 \xrightarrow{\cong} \mathcal{K}_{x_1}^\pi \oplus \mathcal{K}_{x_3}^\pi$$

Furthermore let us fix the following vector space $\mathbb{V} := \mathcal{K}_c^\pi$ to be the stalk of the local system $\mathcal{K}$ at the point $c := \left(0, \pi, \frac{1}{2}, 0\right)$ (which is obviously a point in the fiber $\overline{A} \times \{\pi\}$). By analytic continuation we can identify every non-zero stalk $\mathcal{K}_x^\vartheta$ with $\mathbb{V}$ for all ϑ.
Now consider the following diagram of isomorphisms:

$$\begin{tikzcd}
\mathcal{L}_0 \arrow[rr, "S_0^\pi"] \arrow[d, "\cong"'] & & \mathcal{L}_\pi \arrow[d, "\cong"] \\
H^1\left(\overline{A} \times \{0\}, \overline{\beta}_!^0 \mathcal{K}^0\right) \arrow[rr] \arrow[d, "\Gamma_0"'] & & H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) \arrow[dl, "\widetilde{\Gamma}_\pi"'] \arrow[d, "\Gamma_\pi"] \\
\mathcal{K}_{x_1}^0 \oplus \mathcal{K}_{x_3}^0 \arrow[r, "\mu_0^\pi"] \arrow[d] & \mathcal{K}_{x_1}^\pi \oplus \mathcal{K}_{x_3}^\pi \arrow[r, "\nu_\pi"] \arrow[d] & \mathcal{K}_{x_2}^\pi \oplus \mathcal{K}_{x_4}^\pi \arrow[d] \\
\mathbb{V} \oplus \mathbb{V} \arrow[r, "Id"] & \mathbb{V} \oplus \mathbb{V} \arrow[r, "N_\pi"] & \mathbb{V} \oplus \mathbb{V}
\end{tikzcd}$$

It remains to determine the map ν_π (respectively N_π). Therefore we will combine the coverings $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$ of $\overline{A} \times \{\pi\}$ to a refined covering $\mathfrak{B}$. We get refinement maps $\mathrm{ref}_{\mathfrak{A} \to \mathfrak{B}}$ and $\mathrm{ref}_{\widetilde{\mathfrak{A}} \to \mathfrak{B}}$ and receive the following picture:



Since $\mathfrak{B}$ is again a Leray covering the refinement maps induce isomorphisms on the cohomology groups. Thus we will extend' the above diagram of isomorphisms in the following way:

$$\begin{tikzcd}
 & \check{H}^1\left(\mathfrak{A}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) \arrow[r, "\nu'_\pi"] & \check{H}^1\left(\mathfrak{A}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) & \\
H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}_!^0 \mathcal{K}^0\right) \arrow[r, "\cong"] \arrow[dr, "\widetilde{\Gamma}_\pi"'] & \check{H}^1\left(\widetilde{\mathfrak{A}}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) \arrow[u, "\mathrm{ref}_{\widetilde{\mathfrak{A}} \to \mathfrak{B}}"] \arrow[d, "\cong"'] & \check{H}^1\left(\mathfrak{A}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) \arrow[u, "\mathrm{ref}_{\mathfrak{A} \to \mathfrak{B}}"'] \arrow[d, "\cong"] & H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) \arrow[l, "\cong"'] \arrow[dl, "\Gamma_\pi"] \\
 & \mathcal{K}_{x_1}^\pi \oplus \mathcal{K}_{x_3}^\pi \arrow[r, "\nu_\pi"] \arrow[d, "\cong"'] & \mathcal{K}_{x_2}^\pi \oplus \mathcal{K}_{x_4}^\pi \arrow[d, "\cong"] & \\
 & \mathbb{V} \oplus \mathbb{V} \arrow[r, "N_\pi"] & \mathbb{V} \oplus \mathbb{V} &
\end{tikzcd}$$

To determine the refinement maps on the first cohomology groups we have to consider the Čech complex for $\mathfrak{B}$.

We set the following index sets

- $I := \{1, 2, 3\}$ (indices corresponding to the covering $\mathfrak{A} = \bigcup_{i \in I} A_i$)
- $\widetilde{I} := \{1, 2, 3\}$ (corresponding to $\widetilde{\mathfrak{A}} = \bigcup_{i \in \widetilde{I}} \widetilde{A}_i$)
- $J := \{1, 2, 3, \ldots, 11\}$ (corresponding to $\mathfrak{B} = \bigcup_{j \in J} B_j$)
- $K := I \times J$
- $J' := \{2, 3, 9, 10, 11\} \subset J$
- $K' = \{(1,2), (1,9), (1,11), (2,3), (2,8), (2,9), (2,10), (2,11), (3,9), (3,10), (3,11),$
  $(4,9), (4,10), (4,11), (5,9), (5,10), (6,9), (8,9), (9,10), (10,11)\} \subset K$

The Čech complex for $\mathfrak{B}$ is given by:

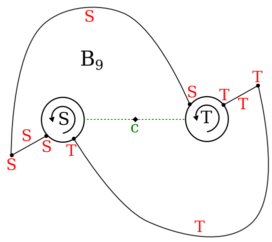
$$\begin{tikzcd}
\check{C}^0 \arrow[r, "d_0"] \arrow[d, equal] & \check{C}^1 \arrow[r, "d_1"] \arrow[d, equal] & \check{C}^2 \arrow[r, "d_2"] & \ldots \\
\bigoplus\limits_{j \in J'} \check{H}^0\left(B_j, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) & \bigoplus\limits_{(i,j) \in K'} \check{H}^0\left(B_i \cup B_j, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) & &
\end{tikzcd}$$

To determine the map d_0 we will use the following identification of $\check{C}^1$:
As above, by analytic continuation we can identify each component $\check{H}^0\left(B_i \cup B_j, \overline{\beta}_!^\pi \mathcal{K}^\pi\right)$ with

$\mathbb{V}^{k(i,j)} := \bigoplus_{l=1}^{k(i,j)} \mathbb{V}$ where $k(i,j)$ denotes the number of connected components of $(B_i \cap B_j)$. $k(i,j) = 1$ for all $(i,j) \in J$ except for $(3,9)$ and $(6,9)$ where it is equal to 2. So we have an isomorphism

$$\check{C}^1 \cong \bigoplus_{(i,j)\in K'} \mathbb{V}^{k(i,j)}$$

Now if we take a section b_9 of B_9, restrict it to a boundary component of B_9 (which is the intersection with one of the bordering B_is) and identify it with $\mathbb{V}$, we have to take care about the monodromies S, T around the two leaks in $\overline{A} \times \{\pi\}$. The following picture shows, by restriction to which boundary component we receive monodromy.



Remark 3.7: If we follow S and T in the coordinates of $\overline{A} \times \{\vartheta\}$, we can also describe them in terms of monodromy around the divisor components: S can be described by a path $\gamma_S : [0,1] \to \overline{A} \times \{\vartheta\}, \tau \mapsto (0, \vartheta, 0, \tau \cdot 2\pi)$ in the coordinates $(|t|, \vartheta, |x|, \theta_x)$. If we look at this path in the (complex) blow up of the singular locus of $\mathcal{M}$, γ_S corresponds to the monodromy around $\widetilde{S}_1$. In the same way T is given by $\gamma_T : [0,1] \to \overline{A} \times \{\vartheta\}, \tau \mapsto (0, \vartheta, 0, \tau \cdot 2\pi)$ in the coordinates $\left(|\tilde{u}'_1|, \vartheta, |\tilde{v}'_1|, \theta_{\tilde{v}'_1}\right)$ and it corresponds to the monodromy around the strict transform of S_1. Note that S and T do not depend on ϑ.

Now one can write down easily the map d_0 of the Čech complex and the refinement maps $\mathrm{ref}_{\mathfrak{A}\to\mathfrak{B}}$ respectively $\mathrm{ref}_{\widetilde{\mathfrak{A}}\to\mathfrak{B}}$ with respect to the above analytic continuation.
Furthermore let (a_2, a_4) be a base of $\check{H}^1\left(\mathfrak{A}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) \cong \mathcal{K}^\pi_{x_2} \oplus \mathcal{K}^\pi_{x_4} \cong \mathbb{V} \oplus \mathbb{V}$ and $(\tilde{a}_1, \tilde{a}_3)$ a base of $\check{H}^1\left(\widetilde{\mathfrak{A}}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) \cong \mathcal{K}^\pi_{x_1} \oplus \mathcal{K}^\pi_{x_3} \cong \mathbb{V} \oplus \mathbb{V}$. We receive two bases of $\check{H}^1\left(\mathfrak{B}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right)$, namely $\mathrm{ref}_{\widetilde{\mathfrak{A}}\to\mathfrak{B}}(\tilde{a}_1, \tilde{a}_3)$ and $\mathrm{ref}_{\mathfrak{A}\to\mathfrak{B}}(a_2, a_4)$. Thus ν' is determined by representing the base $\mathrm{ref}_{\widetilde{\mathfrak{A}}\to\mathfrak{B}}(\tilde{a}_1, \tilde{a}_3)$ in terms of $\mathrm{ref}_{\mathfrak{A}\to\mathfrak{B}}(a_2, a_4)$.
As before we identify $\check{C}^1\left(\mathfrak{B}, \overline{\beta}_!^\pi \mathcal{K}^\pi\right) \cong \bigoplus \mathbb{V}^{k(i,j)}$ and we end up in solving the following equation for each $(i,j) \in K'$:

$$\left(\mathrm{ref}_{\widetilde{\mathfrak{A}}\to\mathfrak{B}}(\tilde{a}_1, \tilde{a}_3)\right)_{(i,j)} = \left(\mathrm{ref}_{\mathfrak{A}\to\mathfrak{B}}(a_2, a_4)\right)_{(i,j)} \mod im\,(d_0)$$

We get the following result:

$$\tilde{a}_1 = -a_2 + \left(1 - ST^{-1}\right) a_4, \;\; \tilde{a}_3 = -ST^{-1} a_4$$

and consequently the map N_π is given by the matrix

$$\begin{pmatrix} -1 & 1 - ST^{-1} \\ 0 & -ST^{-1} \end{pmatrix}$$

For calculating $S^0_\pi : \mathcal{L}_\pi \to \mathcal{L}_0$, we will use exactly the same procedure, except that we have to take care about the continuation to the vector space $\mathbb{V}$.
First we fix another vector space $\mathbb{W} := \mathcal{K}^0_c$ where $c = \left(0, 0, \frac{1}{2}, 0\right) \in \overline{A} \times \{0\}$ and consider the following diagram:

$$\begin{tikzcd}
\mathcal{L}_\pi \arrow[rr, "S^0_\pi"] \arrow[d, "\cong"'] & & \mathcal{L}_0 \arrow[d, "\cong"] \\
H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}^\pi_! \mathcal{K}^\pi\right) \arrow[rr] \arrow[d, "\Gamma_\pi"'] & & H^1\left(\overline{A} \times \{0\}, \overline{\beta}^0_! \mathcal{K}^0\right) \arrow[dl, "\widetilde{\Gamma}_0"'] \arrow[d, "\Gamma_0"] \\
\mathcal{K}^\pi_{x_2} \oplus \mathcal{K}^\pi_{x_4} \arrow[r, "\mu^0_\pi"] & \mathcal{K}^0_{x_2} \oplus \mathcal{K}^0_{x_4} \arrow[r, "\nu_0"] \arrow[d] & \mathcal{K}^0_{x_1} \oplus \mathcal{K}^0_{x_3} \arrow[d] \\
 & \mathbb{W} \oplus \mathbb{W} \arrow[r, "N_0"] & \mathbb{W} \oplus \mathbb{W}
\end{tikzcd}$$

As before, for the determination of the map ν_0 (respectively N_0) we combine the coverings $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$ of $\overline{A} \times \{0\}$ to the refined covering $\mathfrak{B}$ and get the refinement maps $\mathrm{ref}_{\mathfrak{A} \to \mathfrak{B}}$ and $\mathrm{ref}_{\widetilde{\mathfrak{A}} \to \mathfrak{B}}$ which induce isomorphisms on the cohomology groups.
We get:

$$N_0 = \begin{pmatrix} -TS^{-1} & 0 \\ 1 - TS^{-1} & -1 \end{pmatrix}$$

Now we extend ν_0 respectively N_0 to the vector space $\mathbb{V} \oplus \mathbb{V}$ by μ^0_π (which does not affect N_0):

$$\begin{tikzcd}
\mathcal{K}^\pi_{x_2} \oplus \mathcal{K}^\pi_{x_4} \arrow[r, "\mu^0_\pi"] \arrow[d] & \mathcal{K}^0_{x_2} \oplus \mathcal{K}^0_{x_4} \arrow[r, "\nu_0"] \arrow[d] & \mathcal{K}^0_{x_1} \oplus \mathcal{K}^0_{x_3} \arrow[d] & \mathcal{K}^\pi_{x_1} \oplus \mathcal{K}^\pi_{x_3} \arrow[l, "\mu^0_\pi"'] \arrow[d] \\
\mathbb{V} \oplus \mathbb{V} \arrow[r, "Id"] & \mathbb{W} \oplus \mathbb{W} \arrow[r, "N_0"] & \mathbb{W} \oplus \mathbb{W} & \mathbb{V} \oplus \mathbb{V} \arrow[l, "Id"']
\end{tikzcd}$$

Let us fix two isomorphisms Σ_0 and Σ_π, which we will call the standard identification of $H^1\left(\overline{A} \times \{0\}, \overline{\beta}^0_! \mathcal{K}^0\right)$, respectively $H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}^\pi_! \mathcal{K}^\pi\right)$ with the vector space $\mathbb{V} \oplus \mathbb{V}$.

$$\Sigma_0 : H^1\left(\overline{A} \times \{0\}, \overline{\beta}^0_! \mathcal{K}^0\right) \xrightarrow{\Gamma_0} \mathcal{K}^0_{x_1} \oplus \mathcal{K}^0_{x_3} \xrightarrow{\mu^\pi_0} \mathcal{K}^\pi_{x_1} \oplus \mathcal{K}^\pi_{x_3} \to \mathbb{V} \oplus \mathbb{V}$$

$$\Sigma_\pi : H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}^\pi_! \mathcal{K}^\pi\right) \xrightarrow{\Gamma_\pi} \mathcal{K}^\pi_{x_2} \oplus \mathcal{K}^\pi_{x_4} \to \mathbb{V} \oplus \mathbb{V}$$

Summarizing the previous calculations, we can now prove Theorem 1.5:

3.4 Conclusion: Proof of Theorem 1.5

At first remark that $\mu_0^\pi \circ \mu_\pi^0 : \mathcal{K}_x^\pi \oplus \mathcal{K}_y^\pi \to \mathcal{K}_x^0 \oplus \mathcal{K}_y^0 \to \mathcal{K}_x^\pi \oplus \mathcal{K}_y^\pi$ is the isomorphism arising from varying the angel ϑ via the path $\gamma_U : [0,1] \to \mathbb{S}^1, \tau \mapsto \pi + \tau \cdot 2\pi$. This corresponds to the monodromy U around the divisor component $\{0\} \times \mathbb{P}^1$.
Furthermore from Theorem 3.6 we know that

$$\left(H^1\left(\overline{A} \times \{0\}, \beta_!^0 \mathcal{K}^0\right), H^1\left(\overline{A} \times \{\pi\}, \beta_1^\pi \mathcal{K}^\pi\right), \sigma_0^\pi, \sigma_\pi^0\right)$$

defines a set of Stokes data. With the standard identifications of our vector spaces we get the following diagram:

$$\begin{tikzcd}
\mathbb{V} \oplus \mathbb{V} \arrow[rr, "N_\pi"] & & \mathbb{V} \oplus \mathbb{V} \\
\mathcal{K}_{x_1}^\pi \oplus \mathcal{K}_{x_3}^\pi \arrow[u] & & \\
\mathcal{K}_{x_1}^0 \oplus \mathcal{K}_{x_3}^0 \arrow[u, "\mu_0^\pi"'] & & \mathcal{K}_{x_2}^\pi \oplus \mathcal{K}_{x_4}^\pi \arrow[uu] \\
H^1\left(\overline{A} \times \{0\}, \overline{\beta}_!^0 \mathcal{K}^0\right) \arrow[u, "\Gamma_0"'] \arrow[d, "\Gamma_0"] \arrow[uuu, bend left=60, "\Sigma_0"] \arrow[dddd, bend right=60, "\Sigma_0"'] \arrow[rr, shift left, "\sigma_0^\pi"] & & H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}_1^\pi \mathcal{K}^\pi\right) \arrow[ll, shift left, "\sigma_\pi^0"] \arrow[u, "\Gamma_\pi"] \arrow[d, "\Gamma_\pi"'] \arrow[uuu, bend right=60, "\Sigma_\pi"'] \arrow[dddd, bend left=60, "\Sigma_\pi"] \\
\mathcal{K}_{x_1}^0 \oplus \mathcal{K}_{x_3}^0 \arrow[d, "\mu_0^\pi"] & & \mathcal{K}_{x_2}^\pi \oplus \mathcal{K}_{x_4}^\pi \arrow[ddd] \\
\mathcal{K}_{x_1}^\pi \oplus \mathcal{K}_{x_3}^\pi \arrow[dd] & & \\
 & & \\
\mathbb{V} \oplus \mathbb{V} & \mathbb{V} \oplus \mathbb{V} \arrow[l, "\left(\begin{smallmatrix} U & 0 \\ 0 & U \end{smallmatrix}\right)"] & \mathbb{V} \oplus \mathbb{V} \arrow[l, "N_0"]
\end{tikzcd}$$

Since Σ_0 and Σ_π respect the given filtrations of the vector spaces $H^1\left(\overline{A} \times \{0\}, \overline{\beta}_!^0 \mathcal{K}^0\right)$ and $H^1\left(\overline{A} \times \{\pi\}, \overline{\beta}_1^\pi \mathcal{K}^\pi\right)$, it follows that the induced filtrations on $\mathbb{V} \oplus \mathbb{V}$ are mutually opposite with respect to $S_0^1 = \Sigma_\pi \circ \sigma_0^\pi \circ {\Sigma_0}^{-1}$ and $S_1^0 = \Sigma_0 \circ \sigma_\pi^0 \circ {\Sigma_\pi}^{-1}$. Thus we conclude that (L_0, L_1, S_0^1, S_1^0) defines a set of Stokes data for $\mathcal{H}^0 p_+ \left(\mathcal{M} \otimes \mathcal{E}^{\frac{1}{y}}\right)$.

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